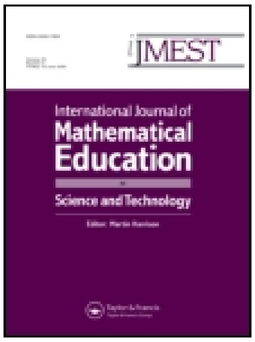

International Journal of Mathematical Education in Science and Technology

ISSN: 0020-739X (Print) 1464-5211 (Online) Journal homepage: http://www.tandfonline.com/loi/tmes20

Understanding the difficulties faced by engineering undergraduates in learning mathematical modelling

Wanmei Soon , Luis Tirtasanjaya Lioe & Brett McInnes

To cite this article: Wanmei Soon , Luis Tirtasanjaya Lioe & Brett McInnes (2011) Understanding the difficulties faced by engineering undergraduates in learning mathematical modelling, International Journal of Mathematical Education in Science and Technology, 42:8, 1023-1039, DOI: 10.1080/0020739X.2011.573867

To link to this article: https://doi.org/10.1080/0020739X.2011.573867

Published online: 03 Jun 2011.

International Journal of Mathematical Education in Science and Technology, Vol. 42, No. 8, 15 December 2011, 1023–1039

Taylor & Francis
Taylor & Francis Group

Understanding the difficulties faced by engineering undergraduates in learning mathematical modelling

Wanmei Soon[a]*, Luis Tirtasanjaya Lioe[b] and Brett McInnes[c]

[a]*Mathematics and Mathematics Education Academic Group, National Institute of Education, Nanyang Technological University, 1 Nanyang Walk, Singapore 637616, Singapore;* [b]*Centre for Research in Pedagogy and Practice, National Institute of Education, Nanyang Technological University, Singapore 637616, Singapore;* [c]*Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543, Singapore*

(*Received 10 February 2010*)

The teaching of mathematics in Singapore continues, in most cases, to follow a traditional model. While this traditional approach has many advantages, it does not always adequately prepare students for University-level mathematics, especially applied mathematics. In particular, it does not cultivate the ability to deal with *non-routine problems*, which is an essential virtue for any practitioner of *mathematical modelling* (MM). Here, we argue that this inability to handle non-routine problems is the principal reason for the serious difficulties experienced by Singaporean students who encounter MM at the tertiary level in Singapore universities. A survey was conducted, primarily to understand the difficulties facing the preliminary batch of first-year undergraduate engineering students doing a course in MM using differential equations (DEs) and linear algebra. Our work is motivated by concerns that the novelty of this course in the Singaporean context could lead to difficulties for these students. Students' abilities in attempting novel modelling-type problems, the techniques they employ in solving such problems, their comments on the course and expectations of their lecturers and tutors are being probed. We present our analyses of the survey results and discuss the implications for future work.

Keywords: mathematical modelling; differential equations; undergraduate engineering

1. Introduction

Singapore is a hierarchical society in which hierarchy is reinforced by an aversion to controversy. As an admittedly broad generalization, one can say that Singaporean students are encouraged to believe that there is precisely one right way of doing things; this high-level normative judgement relieves the students of making such judgements of their own. Following a series of five to seven lessons for six primary classrooms and four secondary classrooms, [1] indicated a strong use of routine questions in the lessons, that is, above 90% (134 out of 138 in primary classrooms, and 31 of 31 questions in secondary classrooms). This indicated very little, if any,

*Corresponding author. Email: wanmei.soon@nie.edu.sg

ISSN 0020–739X print/ISSN 1464–5211 online

http://dx.doi.org/10.1080/0020739X.2011.573867
http://www.tandfonline.com

exposure of students to non-routine and open-ended problems. Anecdotal evidence suggests that such lack of exposure to problems other than closed and routine problems, is typical in the primary, secondary and pre-university levels in Singapore schools. Hence, this leads to the impression of mathematics as the quintessentially 'closed' subject. In fact, this supposed lack of non-routine problems is regarded by many students as one of the principal attractions of mathematics as a course of study, and anecdotal evidence suggests that it is one of the reasons given by students for choosing to study the subject at tertiary level. Mathematics is thought of as being a neat and precise subject, and mainly consists of applications of formulae. This is a regrettable development, and it is a matter of some urgency that it be countered.

The impression of mathematics as the quintessentially 'closed' subject causes serious difficulties when students encounter the concept of *mathematical modelling* (MM).[1] This occurs in the Singapore context somewhat late in the curriculum for most students, but during the *first year of study* for Engineering students. These latter students are of particular interest, because the special difficulties of learning MM are superimposed on the general (in Singapore, often severe) difficulties of making the secondary/tertiary transition. Students' inabilities to *deal with non-routine problems* and their *stereotyped thinking about word problems* [2] certainly do not help them in this transitional period. In particular, the students find it very difficult to make the *normative* judgements essential to the MM process: to many of them, speaking of 'good' or 'bad' models makes no more sense than speaking of 'good' or 'bad' quadratic equations. The discomfort experienced when the student is required to make such judgements and to deal with non-routine problems more generally is, we contend, one of the main reasons for the difficulties students report when asked about their responses to MM-based courses.

We conducted a survey of the preliminary batch of first-year undergraduate engineering students doing a course in MM of differential equations (DEs) and linear algebra. According to some lecturers and tutors of this course, one of the most frequent and characteristic laments of these students is that 'we do not know how to get started' on modelling problems. Often this inability to 'get started' is ascribed to the well-known difficulty of 'converting words to mathematics', since most MM problems are indeed presented verbally in tutorial problems. However, English is the language of education in Singapore from primary level; while an engineering student may have a command of the language which is far from perfect, it seems implausible to suppose that many of them would encounter serious difficulties in actually construing the meaning and general intent of such questions.[2] On the other hand, these students rarely complain about questions of the form 'solve the following DEs', even the hard ones: they have in fact been thoroughly drilled in solution techniques through this course and even at school.

The upshot is that students lack motivation towards such a course, as too much time is needed to be spent on 'getting started' on modelling problems. A frequently given reason for not attempting the modelling-type tutorial problems is that they would not be able to do them even if they tried, so they might as well work on the many other subjects that compete for their limited time. This may be regarded as a particular instance of the phenomenon discussed in [3,4]: the main difficulty students encounter in modelling is the inability to see the links between the mathematics they learn and the real world. Such difficulties are certainly manifested in these Singaporean students who are mostly taught in traditional ways.

Our view is that the problem of 'converting words to mathematics' is merely symptomatic: the presence of 'so many words' is subtly alarming to these students because it suggests a return to realms of *non-routine problems* from which they believed they had escaped by enrolling in engineering. It is telling that a frequent comment students made during the course is that 'we do not know what is wanted', revealing an implicit assumption that only *one* objective could ever possibly be 'wanted'. Their discomfort with problems that do not explicitly state what are required of them is prominent. Even if a percentage of students are able to actually start on a problem, they often fail to complete their tasks due to a lack of confidence and insecurities about whether they are on the right track.

A model of the MM process commonly considered is: 'Real life scenario $\rightarrow$ Mathematical representation $\rightarrow$ Solution of mathematical problem $\rightarrow$ Interpretation of results in real-life'.

We believe that the students involved in our investigation are well versed in the 'Mathematical representation $\rightarrow$ Solution of mathematical problem' stage. These students have been exposed to the mathematical principles required by this study in some pre-university and university courses they took (prior to doing this survey). They include polar coordinates and solution of ordinary differential equations (ODEs). Therefore, they should be proficient in the solution procedures.

We believe that once the students are able to arrive at the mathematical representation of a problem, they will be able to proceed with the solution process. However, the unfamiliarity with the real-life scenario seems to prevent them from getting past the first stage. Although students' difficulties in this area are evident, little research has been done on understanding these difficulties. What are the specific difficulties that students face in translating a real-life problem to a mathematical problem? Do the difficulties involve physics or mathematical concepts, logical thinking, or are they linguistic in nature? This study is our first attempt to answer these questions.

Through our work, we wish to identify any particular approaches these engineering students undertake when attempting modelling-type problems, the key difficulties students face, their understanding of different quantities given or required in problems, their abilities to identify the relationships involved, and the tools needed to solve problems. Their comments on the course and expectations of the lecturers and tutors involved, together with self-perceived difficulties are also sought. Note that there is no intention to generalize our findings to the whole student population. Our hope is that, through this preliminary study, our suggestions can assist students constructively in their learning. In addition, with greater awareness of students' difficulties and more detailed feedback, the educators involved can adjust their teaching to help students learn better.

2. Mathematical modelling

Though developing MM skills is deemed increasingly important worldwide, current practices of teaching DEs are mainly focused on actual solution methods and forms of algebraic answers. This is usually due to time constraints and difficulties associated with the teaching of MM. As for modelling with linear algebra, it is even more rarely touched on in university courses due to limited exposure of educators to such a focus. Much educational research has been done on MM for example, studies of students' working styles in modelling activities, use of technology in MM and assessment of students' attainment at some specific stages of modelling.

In the earlier years when MM courses were newly integrated to some schools' curriculums, investigations have been undertaken to solicit students' methods of solving modelling problems. To understand the different approaches that students take when attempting the same modelling activity, [5] posed a problem which did not require any specialist knowledge and could be tackled by advanced-level mathematics students. The students basically used either an experimental or an analytic approach. The general trend was that the best estimates were obtained when an analytic approach was used, while the experimental approach produced the most varied results. Many students felt that their solutions were realistic and failed to see that their estimations or assumptions were inappropriate. Thus it is important to point out the need of evaluating one's answers when using mathematics to solve real-life problems. Then in the study by Maull and Berry [6] on the working styles of mathematics undergraduate students in solving modelling tasks, students were asked to model the cooling of tea. The authors found that students either conducted experiments or applied known formulae at the initial stages. Some jumped straight into DE models without considering the simplifications needed. The students could not appreciate the importance of making assumptions and seemed unclear about why they were performing certain steps, though they seemed to know what they had to do. There were also students who used inappropriate tools or assumed inappropriate mathematical models. Hence, the facilitators of modelling activities should promote the need for students to stand back initially from the actual problem and spend time studying the physical situation involved. Further investigative studies of students' strategies used and attitudes towards modelling can be found in [7,8].

Technology is often incorporated into MM activities. On top of basic computer tools like excel, the use of graphic calculators (GC) and computer algebra systems (CAS) in modelling have emerged over the years. Jones [9] explored the teaching of modelling using maple (a CAS). In that work, students were given data to fit certain well-known DE models, in order to determine population levels. The CAS allowed a larger group of students to use mathematical ideas since a detailed understanding may not be necessary. However, the disadvantage was that students may be unable to gain an intrinsic understanding of the mathematics involved, and hence lack the ability to judge the correctness of the solutions found. Then in [10], the use of GC in two problems (one structured modelling and one open modelling problem) was discussed. In structured modelling, the context contains a particular unknown to find, and the modeller knows exactly what needs to be done. On the other hand, open modelling involves problem statements in real-world terms (without explicit mathematics), and the modeller has to work through a complete modelling cycle, including formulating the problems mathematically, integrating real-world data and mathematical manipulation. The varied notions (and different categories) of MM that teachers have were also mentioned. Brown concluded that more open modelling assignments are necessary to fully utilize the potential of GC, and the wide variations in the understanding of the term MM cause difficulties to educators.

Verschaffel et al. [11] conducted a study which revealed that instructors are somewhat responsible for pupils' inabilities to activate relevant real world knowledge when dealing with word problems modelling real-life scenarios. In the study, pre-service teachers were asked to solve some school word problems themselves and to analyse pupils' solutions of the same problems. The results showed a strong overall tendency by the student teachers to exclude real-world knowledge and realistic considerations when attempting word problems, and when studying pupils' answers. Thus it is important to focus on teachers' disposition towards modelling. The workshops conducted by Martinez-Luaces [12] to engage secondary and university teachers in modelling were positively received. In the workshops, teachers of different disciplines worked in teams to solve problems posed (taken from papers and textbooks) and to propose new problems. Many proceeded to incorporate these modelling examples in their teaching and to continue working in teams, after the workshops ended.

A peer tutoring scheme was designed by Houston and Lazenbatt [13] to support the learning of MM and several advantages were found. In that scheme, students investigated separate topics in different groups, presented their work to peers in seminars and took exams on the topics presented by all. The students asserted that it was a valuable learning experience, with enhanced knowledge of the topic that they had to work together on (as compared with the case of individual learning). In addition, the students believed that their problem-solving, research, presentation and team-work skills were elevated in the process. Thus working in groups should be beneficial to students. Then in [14], a peer-to-peer assessment scheme, coupled with take-home exams and an MM survey, were used to monitor students' attitudes, skills and competencies in MM. Though a third of the students reacted negatively towards peer-to-peer assessment and most students found MM difficult, the authors believe that students' evaluations of their peers' work encourage them to be more cautious in developing their own mathematical models and bring about changes in their views of mathematics (and MM). Another attempt to improve students' mathematical attitude based on MM was discussed in [15]. Seven stages of MM were proposed therein and students collaborated to work on a given problem through all the stages. A positive attitudinal improvement was obtained as a result.

The papers by Haines and Crouch [16], Rowland and Jovanoski [17], Lege [18], Chaachoua and Saglam [19], Rowland [20] and Klymchuk et al. [21] focused on some specific stages of modelling. Haines and Crouch [16] attempted to identify students' attainment at some particular modelling stages. The students at two universities answered a multiple-choice questionnaire which required them to move between the real world and the mathematical model: the real world problem statement; formulation of a model; and evaluation of a solution. For example, students were asked to consider the problem: What is the best size for pushchair wheels? They were then asked to choose one of the five given clarifying questions which they deemed most addresses the smoothness of the ride as felt by a child. Such a problem was designated to test a student's ability to move from the real world to the mathematical world. The authors reflected that the short questions posed to students helped to identify students' difficulties and levels of conceptual understanding.

In their further work in Crouch and Haines [4], questionnaires were used to analyse the correlations between nine different descriptors of student behaviour according to three viewpoints: an overall perspective; a student's perspective; and a modelling cycle perspective. An interview was also conducted with a tutor of MM courses in a bid to understand the developmental processes through which the learner passes in progressing from novice to expert behaviour. The problems faced by students include a lack of knowledge and inexperience in abstraction in moving from the real to the mathematical world, thus teaching and learning styles need to focus greatly on this stage of modelling.

An inquiry-based approach called the 'modelling discussion' was introduced by Lege [18] to assist students of different abilities in the creation of mathematical models (perhaps the most difficult stage of modelling). Students are asked a series of questions which lead them to build a mathematical model that they believe can solve a given real world problem. Such a scaffolding approach can be very useful for novices in modelling, in building up their confidence to tackle complicated problems, though it can be argued that the cognitive demand of students may be lowered to too great a degree.

Rowland and Jovanoski [17], Chaachoua and Saglam [19], Rowland [20] and Klymchuk et al. [21], focused on modelling with DEs by first year undergraduate students. In the work by Rowland and Jovanoski [17], and Rowland [20], common errors were found in students' (physical) interpretations of different terms in first-order ODEs. In addition, the need for unit consistency in the terms of ODEs was often neglected and many believed that proportionality factors have no units. The authors attributed these difficulties to weak links between mathematics and physical processes, and the typical straightforward problems that students are used to. They emphasized on the need to change the structures of DEs questions used and the way students are being taught. The difficulties in linking physics and mathematics were also investigated in Chaachoua and Saglam [19]. Students usually reduced DEs to algebraic solutions and do not relate them to relevant application fields. Klymchuk et al. [21] discussed the use of non-traditional contexts in MM for engineering students: environment and ecology. In each context, students were tested on their abilities to solve given mathematical models and in further stages of modelling, including the interpretation of solutions and limitations of the models. The authors sought students' opinions of the course and discussed that the positive attitudes students had towards the course contributed significantly to their high performances and the enthusiasm towards the unusual contexts. Hence students' feedback should be considered carefully in the design of curricula for their course.

In the investigations used by the above authors, the mathematical models were typically provided to students to test their understanding of DE use in real physical situations, thus students do not formulate the models themselves. Then in many studies, including those of Graham [5], Jones [9], Houston and Lazenbatt [13] and Maull and Berry [6], though students were required to go through the full modelling cycle, problems which were typical or can be found in the literature were posed to students for them to conduct modelling activities. They reasoned that students would then have minimal difficulties understanding the problems and could dive straight into the modelling process. Our work differs somewhat from them in the types of novel problems posed to students and in the aims of our investigations. However, we also focus on certain stages of modelling, similar to some of the work mentioned above. The details will be discussed in later sections.

3. A description of the modelling course

About 1500 students from the National University of Singapore took the *new* first year undergraduate engineering MM course. They came from a variety of backgrounds, with most of them from the local junior colleges. Their pre-entrance knowledge of calculus was typically non-rigorous. They were often taught how to apply formulae to solve problems, but they had limited exposure to 'word' problems, or modelling-type problems. Thus they were not usually trained to acquire reasoning skills. A small percentage of these engineering students came from polytechnics, where their training in mathematics was of a still less rigorous nature. There were also foreign students from China, Vietnam, India, etc., and some of them have comparatively stronger backgrounds in mathematics than the local students.

In this modelling course, students were taught a variety of topics. They include: basic methods of solving DEs, study of equilibrium and its stability, basic modelling techniques using first order DEs, laplace transform methods, models involving higher order DEs, linear algebra and its applications to Markov modelling and discrete population models, systems of linear first-order equations, phase plane classification and modelling of racial segregation and warfare.

The main objectives of the course were to expose students to real life problems and to inculcate certain modelling skills in students. Though different educators or educational researchers may have varied notions of MM, it is commonly agreed that modelling includes the following stages – the translation of a real scenario to a mathematical model; solving the model and an analysis of the results; and the translation of the mathematical results back to reality. The practice of these stages in modelling by engineering students is deemed important for their future work, hence the course was constructed with these aims in mind.

As mentioned in Falsetti and Rodriguez [15] and Martinez-Luaces [12], students like to solve problems in which they have interest, and which they can relate to. In this course, some problems which relate to recent movies and scenarios that they may experience in their lives are considered. In addition, as one of the intentions of this course is to show how modelling arises in the work of engineers and designers, problems which bring together concepts from different disciplines are often discussed in lectures and in tutorials. While the students may find these problems interesting, it can be a source of distress for most, as reflected by their inabilities to understand and solve such problems in class.

In what follows, we present some of the many problems discussed during the course. The students encountered the first problem within the first few weeks of their course, when they have not yet been taught modelling techniques, but they had learnt how to solve first order DEs. The succeeding question was used to help students set up and analyse population models. The final question requires students to model a type of problem using linear algebra, which was entirely different in context from what they had seen in the lectures.

(1) In very dry regions, the phenomenon called Virga is very important because it can endanger aeroplanes (see http://en.wikipedia.org/wiki/Virga). Virga is rain in air that is so dry that the raindrops evaporate before they can reach the ground. Suppose that the volume of a raindrop is proportional to the 3/2 power of its surface area (why is this reasonable? Note: raindrops are not spherical!). Suppose that the rate of reduction of the volume of a raindrop is proportional to its surface area (why is this reasonable?). Find a formula for the amount of time it takes for a virga raindrop to evaporate completely, expressed in terms of the constants you introduced and the initial surface area of a raindrop. Check that the units of your formula are correct. Suppose somebody suggests that the rate of reduction of the volume of a

raindrop is proportional to the square of the surface area. Argue that this cannot be correct.

(2) Read http://en.wikipedia.org/wiki/Overpopulation. Perhaps Malthus was not so wrong after all? It has been suggested that the Earth's population explosion problem can be solved by sending excess population to colonise other planets. Assuming that a fixed number of colonists are sent out each year, and that the Malthus model would hold if there were no emigration, set up an ODE to describe this plan. Solve it and analyse it (that is, consider various values of your parameters and make predictions, and try to say something interesting about what you find). Next, modify the model by assuming that the rate of emigration is proportional to time (that is, we send more and more people out each year). Analyse!

(3) Miss Lian goes to an Integrated Resort and plays the following game (along with several other players). The players and the croupier each flip coins. If a player's coin matches that of the croupier (both heads or both tails) then the player pays \$1. If they do not match, the house pays the player \$1 (this kind of game is designed to prevent cheating by either party). Initially Miss Lian has \$3. If at any point she loses all her money, she will be violently thrown out of the den with probability 1, and the game goes on without her. If at any point she wins a total of \$2, then also she will be thrown out even more violently with probability 1, because the owner of the gambling den is a crook and does not allow anyone to make more than \$2 from him. What is the probability that Miss Lian will be broke by the time 5 rounds of this game have been played? What is the probability that she will have been thrown out, by 5 rounds, for being too successful?

4. The research method

To assist lecturers and tutors teaching the second and future batches of students doing this course, a survey was conducted a few months after the course was completed by the first batch of students. Our questions were phrased in such a way as to test their understanding of and their abilities to attempt two non-routine problems, to solicit their views of the course and their expectations of the lecturers and tutors involved.

The layout of the survey was as follows.

(1) Moths navigate at night by keeping a fixed angle between their velocity vector and the direction of the Moon (or some bright star; see http://en.wikipedia.org/wiki/Moth). A certain moth flies near to a candle and mistakes it for the Moon. What will happen to the moth? Hint: in polar coordinates (r, θ), the formula for the angle ψ between the radius vector and the velocity vector is given by $\tan(\psi) = r\frac{d\theta}{dr}$ (if you want to derive this formula, remember that the tangential component of velocity is $r\frac{d\theta}{dt}$, where t is time, and the radial component is just $\frac{dr}{dt}$. Now use the chain rule to 'cancel' dt).

(a) Describe the sequence of steps you would take to solve this problem. For example, how do you start working on the problem? Do you have a systematic way of approaching such problems?

(b) (i) What are the known and unknown quantities (variables) in this question?
    (ii) What is the quantity (variable) that you are looking for?

(c) Identify the equations (relationships) involved in this problem.
(d) What are the specific mathematical/physics tools needed to solve this problem?
(e) In your opinion, what are the difficulty levels of this problem in Mathematics, Physics, English and Logical Thinking? (A Likert scale was provided for this question, with the headings – very easy, easy, average, difficult and very difficult).

(2) A fully loaded large oil tanker can be modelled as a solid object with perfectly vertical sides and a perfectly horizontal bottom, so all horizontal cross-sections have the same area, equal to A. Archimedes' principle (http://en.wikipedia.org/wiki/Buoyancy) states that the upward force exerted on a ship by the sea is equal to the weight of the water pushed aside by the ship. Let ρ be the mass density of seawater, and let M be the mass of the ship, so that its weight is Mg, where g is $9.8\,\text{m/s}^2$. When the ship is at rest, find the distance d from sea level to the bottom of the ship. This is called the draught of the ship.

Suppose now that something makes the ship move in the vertical direction. Let $d+x(t)$ be the distance from sea level to the bottom of the ship, where d is the draught as above. Show that, if gravity and buoyancy are the only forces acting on the ship, it will bob up and down with an angular frequency given by $\sqrt{\frac{\rho A g}{M}}$.

Suppose that in fact there is a small amount of friction between the sides of the ship and the seawater, as the ship moves up and down in the sea. The frictional force is equal to $-b\dot{x}$, where b is a constant and $\dot{x}$ is the downward speed of the ship. Furthermore, waves from a storm strike the ship and exert a vertical force $F_0 \cos(\alpha t)$ on the ship, where F_0 is the amplitude of the wave force and α is the wave frequency. Find the most dangerous value of α. Let H be the height of the deck of the ship above sea level when the ship is at rest. For a fixed value of F_0, show how to design a ship that will be able to survive this storm, no matter what α may be.

(a) Describe the sequence of steps you would take to solve this problem. For example, how do you start working on the problem? Do you have a systematic way of approaching such problems?
(b) (i) What are the known and unknown quantities (variables) in this question?
    (ii) What are the quantities (variables) that you are looking for?

(c) Identify the equations (relationships) involved in this problem.
(d) What are the specific mathematical/physics tools needed to solve this problem?
(e) In your opinion, what are the difficulty levels of this problem in Mathematics, Physics, English and Logical Thinking? (A Likert scale was provided for this question, with the headings - very easy, easy, average, difficult and very difficult).

(3) How has this module benefited you? What do you think the lecturers and tutors can do to improve the teaching and learning in this course?

Our choice of the first two test items is in line with the areas of difficulties that we want to study (mainly the translation from real life scenarios to mathematical representations). In the first (moth) problem, a DE was provided to students. The prior knowledge required is polar coordinates and solution of a simple ODE. The question posed in the problem required students to translate the real world to a mathematical world, solve the DE, consider various scenarios and interpret the solution physically. Though the context considered in this problem is unusual and bears no relation to lectures, an attempt to make it less 'abstract' for students was made through the hints given in the problem.

The second problem involves some scaffolding, where questions are posed in relation to successively complicated situations, and students need to practice similar modelling steps to those for the first problem. In addition, students are required to formulate the mathematical models. Note that the students have been taught (in the modelling course and other courses) the concepts needed to solve this problem, that is, Archimedes principle, damped, forced simple harmonic motion (SHM), Hooke's law, Newton's laws, resonance and second order ODEs. In fact, this problem is extremely similar to what has been done in the lectures and tutorials, except that the context is different.

The main instructor of the modelling course had carefully constructed these problems, taking into consideration that students had prior knowledge of the principles required. We quote his motivations behind the problems:

> '(*The moth problem*): *this question shows that you can use mathematics to answer questions which APPARENTLY HAVE NOTHING TO DO WITH MATHS. The point is: we want them to see that maths can be used not just in situations where it is obviously going to be useful* (*physics*, *population dynamics*) *but even in situations that appear to be completely non-mathematical*.'
>
> '(*The tanker problem*): *one crucial aspect of modelling is realising that the SAME mathematics can be used in VERY DIFFERENT situations. A famous example of this is in electrical circuit theory: the equations you get are identical to the equations of a forced damped harmonic oscillator. So you don't need to solve the equations all over again: you just need to map the problem to a mathematical system that you have already studied. Similarly the tanker problem need not involve solving any equations at all: all you have to do is realise that the situation is MATHEMATICALLY identical to what you saw in the lectures, and you just have to map the notation over to the new situation. As I said, it is extremely important in modelling to recognise that only a relatively small number of distinct o.d.e.s actually arise: the same mathematics gets re-used over and over. The logistic equation is another example of this*.'

We called for open participation in our survey as this was a pilot (and small-scale) study. About 50 students responded. The answers given by the students were collated and sorted into different categories. These categories were selected only after the survey was completed, to assist us in doing a qualitative analysis of their responses, including the common errors made, their difficulties and their views.

5. Analysis of survey results

In this section, we will discuss the students' responses in parts according to the questions listed above.

1(a) In attempting the moth problem, we found that about half the students considered drawing a diagram for better understanding. A few students recognized the importance of first understanding the physical situation and what needs to be found exactly. However, it seems that about half of the students believed in diving straight into using the formula provided or finding the equation that represents a problem. No particular systematic approach was discussed. In addition, a very small minority of students actually considered all the three possible scenarios for the moth's behaviour.

1(b)(i) A very small proportion of students identified correctly that ψ is the only known variable in this problem, while r and θ are unknown. Only one student stated that v is unknown (which is true). Many either identified only the known variable or interchanged the known and unknown variables. Some students thought that all the variables were unknown. Relations between the quantities ψ, r and v were even considered as variables. Thus many students cannot understand what is meant by known/unknown quantities. Some seem to think of a variable as being unknown as long as no actual value is assigned to it. However, the concept of 'known and unknown' quantities is important. We believe that it is precisely because students are not able to understand this basic concept that they are unable to start on finding what is needed in the problem.

(ii) Only a few students were able to understand that r is the quantity to be found (in terms of θ). Many students did not answer this part of the survey question. The rest who provided inappropriate responses either thought that time-dependent quantities, for example, $r(t)$ and $\theta(t)$, or the rates $\frac{\mathrm{d}r}{\mathrm{d}t}$ and $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ were needed. They failed to recognize t as an implicit variable and $\theta(t)$ as the independent variable. Therefore, it seems that students have difficulties with the basic concept of classifications of variables as implicit, independent or dependent variables. In addition, a few seem to link how 'fast' the moth moves with the death of the moth.

1(c) Several relationships between different quantities were discussed by students. The equation $\tan(\psi) = r\frac{\mathrm{d}\theta}{\mathrm{d}r}$ was only correctly provided by 15% of the students, though it was explicitly stated in the problem. A few students considered the final answer for r in terms of $\theta(t)$ and ψ to be the equation involved, which is in some sense correct. However, a few students also responded with the relationship between r, v and ψ, though there is no equation linking them. The rest of the students did not respond to this part. Thus it seems that students have a vague idea of the types of relations useful in a problem.

1(d) The DE involved in the problem can be simply solved using 'separation of variables', which was identified by about 11% of the students. Three students stated integration as the tool involved, which is reasonable too. It was interesting to note the variety of responses given by students. They included trigonometry, kinematics and laplace transform (which are not tools needed to solve the problem). Hence, even in a problem where an equation is given explicitly to students, many are still unable to apply known techniques to solve it. This can be due to the lack of true understanding of the problem posed, knowledge of the actual dependent and independent variables and weaknesses in using DE solution techniques.

1(e) The majority of respondents stated that the levels of logical thinking and mathematics involved in the moth problem were difficult, with an average to difficult level for the physics involved, and an average level of english was required. According to a lecturer of this course, the mathematics involved should be easy for students, since it basically consists of a simple 'separation of variables' technique and knowledge of polar coordinates. Only simple logical reasoning is needed here. In addition, since they are all engineering students and should be equipped with sufficient knowledge of kinematics (students have taken an undergraduate physics course with this component, and they could also refer to the website given if necessary), they should be able to understand the moth problem without much difficulty. Thus it is interesting to note the differences between the students' perception of their own levels of difficulties in this problem and a lecturer's perception of students' difficulties.

2(a) With respect to the tanker problem, though more than half the students understood the importance of drawing a diagram and finding out what the question is asking for exactly, most also answered this part of the survey question with responses more relevant to parts (b) and (d). For example, many wrote that Archimedes principle should be applied to obtain d, and SHM is needed to find the frequency ω. Half the students identified the need to find the equations required to represent the problem, with some students stating the need to relate this problem to lecture notes. Thus it seems that when the students are confronted with a problem involving similar principles to those seen in the lectures, many were able to recall some relevant concepts to be applied to the problem.

2(b)(i) The known quantities in this problem are A, M, ρ, g, b, F_0 and H, while the unknown quantities are d, $x(t), \dot{x}$ and α. Barely five students were able to identify all these quantities correctly. Many students either stated only a few of the unknown quantities correctly, or identified some known quantities as unknown quantities and vice versa. Three students gave ω as an unknown quantity, which is in some sense reasonable, since students were required to verify the expression for ω in the problem. The responses here seem to reflect those in 1(b)(i). In addition, though g is clearly a known quantity, many students did not state it.

(ii) Though one would expect students to provide d, the angular frequency and the most dangerous value of α as the most common response, it is surprising that only five students wrote all these. Some scaffolding had been provided in the problem posed, with explicit questions asking for mathematical quantities. Thus it seems that the scaffolding did not help students as expected. However, three students were able to explain correctly that to find the most dangerous value of α, the amplitude of oscillation needs to be found, and the maximum amplitude has to be smaller than H for a ship to be safe. Many students skipped this part of the survey. As compared with the responses to 1(b)(ii), we observe that though no wrong quantities were identified here, most of the students have vague ideas of the quantities required in the problem.

2(c) The equation relevant to the first part of the problem was identified by only five students. Though the required balance of forces was explained explicitly in the problem, it is quite surprising that so few students could do this simplest section of the problem. Only four students could obtain the equation corresponding to the second part, and only two students stated the SHM equation involving ω explicitly. About 15% of students vaguely explained the types of equations needed, for example, SHM, Newton's laws, DEs. However, three students managed to identify the damped, forced oscillation equation relevant to the third part of the problem, while three others tried and obtained wrong equations (either leaving out the quantity A or using wrong signs). Hence, despite the fact that many students could recall certain principles relevant to this problem, most are unable to decipher the useful relations necessary to solve the problem. It seems that students rely heavily on memory instead of true understanding of the concepts.

2(d) The common tools discussed by the students were SHM, Newton's laws, Archimedes principle and solution techniques of DEs. It is interesting to note that most of the students were able to provide some appropriate tools required by the problem, despite them being unable to identify the required quantities or equations. As mentioned above, perhaps much memory work is involved here.

2(e) The self-perceived levels of difficulties faced by most of the students in the tanker problem were: difficult for mathematics, average to difficult for logical thinking and physics and an average level for english. The main lecturer for the course believed that the physics involved in this problem was not difficult, since basically only Archimedes' principle, Hooke's law and Newton's laws were needed (which students should have encountered even in pre-university level courses). The main difficulty for the students should be in mathematics, as students may not understand the dependence of amplitude on α. They probably would expect the amplitude (among other quantities) to depend on time instead. In addition, the solution techniques of the DEs are non-trivial. However, the responses to the previous parts seem to suggest insufficient understanding of the required basic physics concepts, so many may not even be able to arrive at the equations involved, not to mention the amplitude of x.

From students' responses to survey question 3, we see that their attitudes towards the course as reflected in the survey match the reactions that some lecturers and tutors faced during the course. Many students complained that the problems set were too wordy and lengthy, hence they were not able to understand them. They discussed the need for more real-life examples in the lectures as they need to see 'similar' type of modelling problems in lectures before they can do the tutorial problems. There is a need for lecturers and tutors to help students focus more on visualization and the principles behind the problems. Students commented that they feel insecure as they do not know how to study for exams in modelling contexts. These responses point towards the strong reliance of students on the lecturers and tutors, their need for problems which are more 'standard' and do not require deep thinking for understanding and their exam-oriented attitudes. A few students also asked for more interactive tutorials to be conducted, which is not a common phenomenon in this course.

There were also discussions of the benefits obtained from this course. Some students indicated that a strong point was the use of appropriate real-life examples to help students understand the reason for learning the theories depicted in the course. In addition, the link between mathematical models and theories with reality is well-depicted, which prepares students for the field of engineering. Some students appreciated that this course promoted independent learning, improved their analytical skills and logical thinking. One student said that such a mathematical course is better than the usual courses involving mundane, boring and typical mathematical questions.

6. Conclusion

Our findings seem to indicate that students have problems seeing the connection between 'real life contexts' and 'mathematical representations'. This should not be understood as a disaster. Rather, it should be seen as a valuable window to unravel the black box of students' challenges in their thinking process during the 'translation' stage. Indeed, our work reveals students' lack of systematic approaches in attempting problems in modelling contexts and their areas of difficulties.

It is straightforward to develop 'approaches to problem-solving' when teaching MM. For example, the main lecturer of this course recommended to students the following steps:

(a) Begin by drawing a diagram;
(b) Try to separate parameters and variables which are under the control of the modeller from those which are not;
(c) Try to solve for the controllable parameters and variables in terms of the other parameters and variables;
(d) Always check that the units involved are correct;
(e) Examine the DEs to find 'obvious' solutions (which can be obtained by inspection); these invariably have some important meaning, e.g. as equilibrium solutions;
(f) Always subject the final solution to probing questions of the form: 'does this answer make sense?'. The analysis of units in (d) will usually be relevant here.

These recommendations differ somewhat from the steps proposed by some in the past to tackle modelling-type problems (see, e.g. Maull and Berry [6], Crouch and Haines [4] and Falsetti and Rodriguez [15]). These are specially aimed at modelling with DEs. However, our survey and the observations that some lecturers and tutors made during the course strongly suggest that a mere problem-solving 'template', while necessary, is far from sufficient for adequate teaching of MM. Students are not inclined towards following the approaches suggested above, though some do draw diagrams to visualize the MM problems. They seem to have limited patience in reading word problems and are eager to jump straight into the procedures needed to solve them. Thus there is a need to train tutors to execute the above steps explicitly during their discussions with students, and to reflect on the effectiveness of these steps.

One basic issue was the classifications of variables/quantities as known or unknown, implicit or explicit, independent or dependent variables. An alarmingly poor understanding was found, which shows a lack of exposure to the process of quantities identification. Without such basic knowledge, it is difficult for students to start on modelling-type problems and identify the quantity (or quantities) to be found. Though scaffolding is provided in the tanker problem, with questions explicitly stating the required mathematical quantities, many students were still unable to extract the quantities needed. Another obvious problem that the respondents have, is the failure to relate the important quantities to form useful relations. Though for the tanker question, many students could recall relevant concepts learnt through the course, most were still unable to decipher the equations needed. This could be attributed to a heavy reliance on memory or a lack of real understanding of the required mathematical and physics concepts. Even when the DE is provided explicitly in the moth problem, as it is embedded within a modelling context, students can fail to solve it due to misidentification of the different classes of variables. Therefore, it is of utmost importance not to neglect the students' poor abilities in variable classifications, and to assist students in the building of useful relations required by problems.

Even if students can 'solve the problem', many students are left with an uneasy feeling that they have not really done what is expected of a good exponent of MM; and usually they are right in feeling so. We argue that *an ability to tolerate non-routine problems* must be inculcated in students right from the outset of the teaching. In particular, we urge that beginners should *not* be presented with a succession of (necessarily over-simplified) problems with clear-cut 'correct' solutions, before moving on to more realistic situations. Doing so leaves students with the impression that their failure to grasp the subject is due to their inadequate command of language or of technical tools such as those from physics.

The key point here is that MM is fundamentally a *process*, a process in which one begins with admittedly oversimplified and unrealistic models and gradually works towards greater sophistication and realism. Furthermore, this process is *at every step* 'indeterminate': there may be *many* good ways to proceed, right from the outset. We stress that this indeterminacy arises even in the simplest, least realistic models. We would therefore be misleading the students if we encouraged them to believe that 'only hard problems are non-routine'.

Instead we propose that even the simplest questions can be proposed in a manner which compels the student to accept, *from the outset*, that indeterminacy is intrinsic to the whole modelling process. We should *not* ask the students to analyse various possible interpretations of their results only when we move towards more complex examples. In particular, we should, from the outset, formulate questions in such a way that it is clear that there need not be a single 'correct' answer. Though this approach causes some distress, but we argue that in the long run it is better that the students be clear on this point when the problems are still simple, and not try to absorb it at the same time when they are struggling to absorb high-level techniques and concepts from other fields.

As a concrete example: in one tutorial question, the students in this course were guided to set up a model of learning. It was postulated that, in the course of university studies, a given student's overall problem-solving ability might be expected to rise, though of course the function representing this ability is bounded. The students are asked to propose examples of functions which might suit this purpose. Some students suggested rational functions (on suitable domains), others proposed using the hyperbolic tangent. It was impressed on the students that, in view of the lack of precision of the entire model, *both* choices are satisfactory, despite the fact that, mathematically, these functions are very different. This example shows that, even at the very simplest level of model-building, and even when the question involves very elementary mathematics, the intrinsic *indeterminacy* of the modelling process can be brought to the fore.

Acknowledgements

The authors are grateful to the NUS students who participated in the survey and to their tutors for useful feedback.

Notes

1. In recent years there has been preliminary attempts to teach MM in a few Singaporean Junior Colleges. This is a recent phenomenon however, and only selected groups of students may do such a course.

2. It should however be borne in mind that Singapore's universities do take in students from non-anglophone sources; our observations suggest, however, that these students experience no greater difficulties with this aspect of these courses than the 'local' students; reinforcing the contention that the difficulties are *not* primarily linguistic.

References

[1] J.G. Hedberg, K.Y. Wong, K.F. Ho, L.T. Lioe, and Y.S.J. Tiong, Developing the repertoire of heuristics for mathematical problem solving, First Technical Report for Project CRP38/03 TSK, Centre for Research in Pedagogy and Practice, National Institute of Education, Nanyang Technological University, Singapore, 2005.
[2] P.Y. Foong and P.L. Koay, *School word problems and stereotyped thinking*, Teach. Learn. 8(1) (1997), pp. 73–82.
[3] J. Boaler, *Mathematical modelling and new theories of learning*, Teach. Math. Appl. 20(3) (2001), pp. 121–128.
[4] R. Crouch and C. Haines, *Mathematical modelling: Transitions between the real world and the mathematical model*, Int. J. Math. Educ. Sci. Technol. 35(2) (2004), pp. 197–206.
[5] T. Graham, *The ways in which different students respond to the same mathematical modelling problem*, Teach. Math. Appl. 16(1) (1997), pp. 19–22.
[6] W. Maull and J. Berry, *An investigation of student working styles in a mathematical modelling activity*, Teach. Math. Appl. 20(2) (2001), pp. 78–88.
[7] T. Lingefjärd, *Teaching and assessing mathematical modelling*, Teach. Math. Appl. 21(2) (2002), pp. 75–83.
[8] O.R. Jacobini and M.L.L. Wodewotzki, *Mathematical modelling: A path to political reflection in the mathematics class*, Teach. Math. Appl. 25(1) (2006), pp. 33–42.
[9] M. Jones, *Teaching mathematical modelling*, Int. J. Math. Educ. Sci. Technol. 28(4) (1997), pp. 553–560.
[10] R. Brown, *Mathematical modelling in the international baccalaureate, teacher beliefs and technology usage*, Teach. Math. Appl. 21(2) (2002), pp. 67–74.
[11] L. Verschaffel, E. De Corte, and I. Borghart, *Pre-service teachers' conceptions and beliefs about the role of real-world knowledge in mathematical modelling of school word problems*, Learn. Instruction 7(4) (1997), pp. 339–359.
[12] V. Martinez-Luaces, *Engaging secondary school and university teachers in modelling: Some experiences in South American countries*, Int. J. Math. Educ. Sci. Technol. 36(2–3) (2005), pp. 193–205.
[13] S.K. Houston and A. Lazenbatt, *Peer tutoring in a modelling course*, Innov. Educ. Teach. Int. 36(1) (1999), pp. 71–79.
[14] T. Lingefjärd and M. Holmquist, *To assess students' attitudes, skills and competencies in mathematical modelling*, Teach. Math. Appl. 24(2–3) (2005), pp. 123–133.
[15] M. Falsetti and M. Rodriguez, *A proposal for improving students' mathematical attitude based on mathematical modelling*, Teach. Math. Appl. 24(1) (2005), pp. 14–28.
[16] C. Haines and R. Crouch, *Recognizing constructs within mathematical modelling*, Teach. Math. Appl. 20 (2001), pp. 129–138.
[17] D.R. Rowland and Z. Jovanoski, *Student interpretations of the terms in first-order ordinary differential equations in modelling contexts*, Int. J. Math. Educ. Sci. Technol. 35(4) (2004), pp. 503–516.
[18] J. Lege, *Socrates meets the 21st century*, Teach. Math. Appl. 24(1) (2005), pp. 29–36.
[19] H. Chaachoua and A. Saglam, *Modelling by differential equations*, Teach. Math. Appl. 25(1) (2006), pp. 15–22.

[20] D.R. Rowland, *Student difficulties with units in differential equations in modelling contexts*, Int. J. Math. Educ. Sci. Technol. 37(5) (2006), pp. 553–558.
[21] S. Klymchuk, T. Zverkova, N. Gruenwald, and G. Sauerbier, *Increasing engineering students awareness to environment through innovative teaching of mathematical modelling*, Teach. Math. Appl. 27(3) (2008), pp. 123–130.